ALGEBRA REVIEW

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1. Algebraic properties of equations

Our first step during our review will be to recall various rules about manipulating equations. Hopefully these properties will be familiar, even if they might be a bit dusty.

The following is a list of things which is *always* true:

$$a + b = b + a \tag{1}$$

$$a \cdot b = b \cdot a \tag{2}$$

$$a(b + c) = ab + ac \tag{3}$$

$$(b + c)a = ba + ca. \tag{4}$$

People typically remember the first two properties but sometimes forget the last two. Properties (1) and (2) are called the *commutative properties*, while properties (3) and (4) are called the *distributive properties*. It is important to note that (1),(2),(3), and (4) work for *any* numbers, and work for variables as well. (You'll note that in the introduction I didn't say if a, b, c were numbers, variables, or some mixture. That's because it doesn't matter.)

Why these properties hold is a consequence of how addition and multiplication are defined. In principle we could have come up with strange alternative definitions, say $\oplus$ and $\odot$, which did not satisfy these properties. Other branches of math deal with these cases, but we won't be concerned with them here.

For now let's take a look at some examples of (3) and (4) in action. For example, the following are both true:

$$\begin{aligned} 3x + 4x &= (3 + 4)x = 7x, \\ 3.2x + 2.3x &= (3.2 + 2.3)x = 5.5x. \end{aligned}$$

Since these properties are supposed to be elementary, we don't have any practice problems for them.

Let's change gears and talk about equations now. When you see an expression like

$$45x + 35 = 2x - 35$$

it means that the numbers on both sides of the equation are equal. When do things are equal, if we change them in some way – but change them both in the *same* way – the results will also be equal. This fact rests beneath just about everything in math, and is called the *well-defined property.*

So given an equation, we are allowed to *modify* the equation any way we like, provided we make the same changes on the left as we do on the right. This fact is ultimately what allows us to solve equations, as we'll see in the next section.

Sometimes the modifications are simple, like "add 7 to both sides." Other times they can be more complicated. For instance, if we have *two* equations

$$\begin{aligned} 9x + 2y &= 7 \\ -2y &= 4 \end{aligned}$$

we can modify the first equation by "adding 4 to both sides" in a goofy way. Since $-2y = 4$, on one side of the equation we'll just add 4, while on the other side we'll add $-2y$. After all, $4 = -2y$, so we really are "doing the same thing to both sides." Here's what it looks like:

$$\begin{aligned} (4) + 9x + 2y = 7 + (4) &\qquad \text{add 4 to both sides,} \\ (-2y) + 9x + 2y = 7 + 4 &\qquad \text{note that } 4 = -2y, \\ 9x = 11 &\qquad \text{simplify.} \end{aligned}$$

Lastly we have some terminology for the different things that appear in an equation:

$$\overbrace{\underbrace{45}_{\text{coefficient}}\ \underbrace{x}_{\text{variable}}}^{\text{left hand side}} = \overbrace{\underbrace{1274}_{\text{constant}}}^{\text{right hand side}}$$

2. Solving linear equations

Linear equations are those for which the unknown variable appears without any powers (other than 1). For instance, the equation

$$9x + 2 = -\frac{3x}{4}$$

is a linear equation, while the equations

$$7x^2 + 9 = x \text{ and } \frac{12}{x} + x = 4$$

are not.

Most of the algebra in this class is concerned with solving linear equations. When one is presented with a linear equation, it can always be solved using the following process:

(1) Collect all of the numbers on one side.
(2) Collect all instances of the variable we are solving for on the other side.
(3) Simplify.
(4) Divide out by our variable's coefficient.

Here's an example:

$$\begin{aligned}
13x + 5 &= -4x + 7 \\
(-5) + 13x + 5 &= -4x + 7 + (-5) \\
13x &= -4x - 2 \\
(4x) + 13x &= -4x - 2 + (4x) \\
17x &= -2 \\
x &= \frac{-2}{17}.
\end{aligned}$$

We can even perform this process if the equation has more than one variable in it, though we won't wind up with a numeric answer (the answer will just define one of the variables *in terms of* the other, as they say.) In this case, we pick a variable to solve for (y) and treat the other variables (x and z) as if they were numbers:

$$\begin{aligned}
4x - 2y + z &= x - 4y \\
(-4x) + 4x - 2y + z &= x - 4y + (-4x) \\
-2y + z &= -3x - 4y \\
(-z) + -2y + z &= -3x - 4y + (-z) \\
-2y &= -3x - 4y - z \\
(4y) + (-2)y &= -3x - 4y - z + (4y) \\
2y &= -3x - z \\
y &= \frac{-3x - z}{2}.
\end{aligned}$$

3. Solving pairs of linear equations

Sometimes we are given two linear equations which use the same two variables. In this case we call the pair of equations a *system of equations*, and we can use the pair of equations to get numeric values for *both* variables.

Here's a quick example. Suppose we are given the system

$$\begin{aligned} 4x + 2y &= 7, \\ 3x - 2y &= 0. \end{aligned}$$

It is at this point that we make the "clever observation[1]" that, if we just add the equations together, the y's will cancel out:

$$\begin{aligned} (3x - 2y) + (4x + 2y) &= (7) + (0), \\ 7x &= 7 \\ x &= 1. \end{aligned}$$

This allowed us to solve for x. We can now take this value of x and plug it into one of the original equations (it doesn't matter which one) and then solve for y:

$$\begin{aligned} 4x + 2y &= 7 \\ 4(1) + 2y &= 7 \\ 4 + 2y &= 7 \\ (-4) + 4 + 2y &= 7 + (-4) \\ 2y &= 3 \\ y &= \frac{3}{2}. \end{aligned}$$

This trick will always work, though you might need to be smart about it. For instance, consider the system

$$\begin{aligned} 12x + 3y &= 14, \\ 4x + 4y &= 12. \end{aligned}$$

Here if we just add the equations together nothing will cancel out. Luckily, we are able to "modify" one of the equations in such a way that things will work out. Since $12 = 3 \times 4$, we'll multiply the second equation by 3, so that our system becomes

$$\begin{aligned} 12x + 3y &= 14, \\ 12x + 12y &= 36. \end{aligned}$$

[1]Math is all about these.

We can now subtract one equation from the other, and the x's will cancel out:

$$\begin{aligned}(12x+12y)-(12x+3y) &= 14-36\\ 12x-12x+12y-3y &= -22\\ 9y &= -22\\ y &= -\frac{22}{9}.\end{aligned}$$

We can now plug this value of y into (any) of the original equations and solve for x:

$$\begin{aligned}4x+4y &= 12\\ 4x+4\left(-\frac{22}{9}\right) &= 12\\ 4x-\frac{88}{9} &= 12\\ \left(\frac{88}{9}\right)+4x-\frac{88}{9} &= 12+\left(\frac{88}{9}\right)\\ 4x &= 12+\frac{88}{9}\\ 4x &= \frac{12\cdot 9}{9}+\frac{88}{9}\\ 4x &= \frac{108}{9}+\frac{88}{9}\\ 4x &= \frac{196}{9}\\ x &= \frac{196}{9\cdot 4}\\ x &= \frac{49}{9}.\end{aligned}$$

This type of process is guaranteed to work unless one of the following is true:

- The two equations are the same equation, or (more generally) if one of the equations is just some number times the other equation. For instance, you couldn't use the above to solve the system

  $$\begin{aligned}4x+2y &= 9\\ 8x+4y &= 18\end{aligned}$$

  since the second equation is just $2\times$ (the first equation). If you tried the above method, you'd wind up with an equation that says $0=0$, which doesn't help you at all.

- The equations are incompatible. For instance, there just isn't any way to solve the system

$$\begin{aligned} 3x + 2y &= 4 \\ 3x + 2y &= 7. \end{aligned}$$

  In this case, the equations say that $3x + 2y$ is equal to *both* 4 and 7. But this can't be true, because then $4 = 7$, which is false.

These two situations are the only two where the above method won't work. In fact, in these situations *no* method can possibly work. In our class we won't be dealing with these cases.

4. Quadratic equations

We will now review the case where you want to solve an equation of the form

$$ax^2 + bx + c = 0,$$

where a, b, c are all numbers. Such an equation is called a *quadratic equation*, and we can solve for x using the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note the "$\pm$" in the quadratic equation. This means "plus or minus," which is just short hand for saying that the equation has *two* solutions:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Of course, it is possible that these two numbers will be the same (this happens precisely when $b^2 = 4ac$). It is also possible that neither of these numbers is *real*, in the sense that $b^2 - 4ac$ might be a negative number, and "you can't take the square root of a negative number."

The quadratic formula is a fantastic tool. It is a lightsaber: when used properly, it completely hacks your problems to bits. Know it, use it.

Let's look at an example. Here's an equation that we wish to solve:

$$x^2 - x - 12 = 0.$$

The solution requires no mental energy whatsoever: we just plug the numbers into the quadratic equation. The solutions are

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-12)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1+48}}{2} \\ &= \frac{1 \pm \sqrt{49}}{2} \\ &= \frac{1 \pm 7}{2} \\ &= \frac{1+7}{2} \text{ or } \frac{1-7}{2} \\ &= \frac{8}{2} \text{ or } \frac{-6}{2} \\ &= 4 \text{ or } -3. \end{aligned}$$

An idea related to all of this is *factoring*. Here the goal is to find a way to write a quadratic as a product of two linears. For instance,

$$x^2 - x - 12 = (x-4)(x+3).$$

Notice here that the quadratic on the left hand side is precisely the quadratic we solved for in our previous example. Also notice that the numbers -4 and 3 were used on the right hand side, and that these numbers are the negatives of the solutions to our equation (the solutions to our equation were 4 and -3.) This is not a coincidence: it will happen every time.

5. Logarithms

The last type of equation we will consider is that of the form

$$a^x = b,$$

where a, b are numbers. In general, there is no paper-and-pencil way to solve these; sometimes you can look at the numbers and figure out the answer, but sometimes you can't. For instance, if you know your powers of three, you might be able to see the equations

$$3^x = 81$$

and just *know* that $x = 4$. But if you don't already know, off the top of your head, that $3^4 = 81$, then you're seemingly out of luck.

Still, mathematicians and scientists need to work with equations of this form quite often, so they have developed a *notation* for the solution. They *define* $log_a(b)$ to be the number such that

$$a^{log_a(b)} = b.$$

We can think of log as a function that takes two numbers. When I write

$$log_a(b),$$

the number a is called the *base*, while the number b is called the *argument*. Pick any two numbers a, b, and $\log_a(b)$ is just another number which happens to satisfy the relationship $a^{log_a(b)} = b$.

This may seem quite abstract, but as it happens there are two important things which make it useful:

(1) Methods developed in calculus have given us ways to approximately compute $\log_a(b)$ for any choice of a and b up to arbitrary precision. In practice, this means that your calculator can compute $\log_a(b)$ for you. Of course, your calculator is only computing an approximation – sometimes the approximation is exactly correct, sometimes it is off by a small amount. It will certainly be good enough for our purposes.
(2) Based on how $\log_a(b)$ is defined, there are several rules for how we can simplify equations that have log's in them. These rules will be our key to solving equations like $a^x = b$.

Here are some of the properties that logarithms satisfy:

$$\log_a(b) + \log_a(c) = \log_a(b \cdot c), \tag{5}$$

$$\log_a(b) - \log_a(c) = \log_a\left(\frac{b}{c}\right), \tag{6}$$

$$\log_a(b^c) = c \cdot \log_a(b), \tag{7}$$

$$\frac{\log_a(b)}{\log_a(c)} = \log_c(b), \tag{8}$$

$$a^{\log_a(b)} = b, \tag{9}$$

$$\log_a(a) = 1, \tag{10}$$

$$\log_a(a^b) = b. \tag{11}$$

Every one of these properties is a direct consequence of how the logarithm is defined, and can be proved using only the definition as a starting point.

Before we move onto an example, we establish two conventions: when I write $\log(b)$ without a base, I implicitly mean $\log_{10}(b)$. When I write $\ln(b)$, I implicitly mean $\log_e(b)$ (we call $\ln(b)$ the "natural log of b.")

Let's look at an example of how logarithms are used in practice. This example is absolutely typical of how we will be using logarithms. Suppose I want to solve the equation

$$3 = 2^{(x/3)}.$$

I'll now solve this equation for x, using only the properties of logarithms listed above, and a calculator to compute the logarithms at the end:

$3 = 2^{(x/3)}$	Let's solve this for x
$\log(3) = \log\left(2^{(x/3)}\right)$	First I take the log of both sides
$\log(3) = (x/3)\log(2)$	By property (7) I can bring the $(x/3)$ out
$\dfrac{\log(3)}{\log(2)} = x/3$	Now I just move $\log(2)$ to the other side
$\dfrac{3\log(3)}{\log(2)}$	And then bring the 3 to the other side
$\dfrac{3 \cdot 0.477121255}{0.301029996} = x$	Use my calculator to compute $\log(3)$ and $\log(2)$
$4.7548875 = x$	Simplify and I'm done.

As mentioned before, this is typical of how we will use logs. When you have an equation where some number if raised to the x, just do the following:

(1) Rewrite the equation so that it looks like

$$\text{some number} = \big(\text{some number}\big)^{\text{(something with an } x \text{ in it)}}.$$

(2) Take the log of both sides to get

$$\log\big(\text{some number}\big) = \log\Big(\big(\text{some number}\big)^{\text{(something with an } x \text{ in it)}}\Big).$$

(3) Use property (7) to bring the power down:

$$\log\big(\text{some number}\big) = \big((\text{something with an } x \text{ in it}\big)\log\big(\text{some number}\big).$$

(4) Now solve for x, noting that both instances of $\log\big(\text{some number}\big)$ is just some number.