SETS, VENN DIAGRAMS, AND ARGUMENTS

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1. Sets

More than anything else, mathematics is about the process of identifying abstract structures and studying their properties. As students, *Numbers* are the first structures we see. First we see the counting numbers $(0, 1, 2, 3, ..)$, then we learn about the negative numbers $(... -3, -2, -1)$, then fractions, and later we even learn some numbers which aren't even fractions (such as π and e). As we advance, we learn about more sophisticated structures, such as functions, probabilities, and so forth.

In principle, a mathematician *could* grab a particular mathematical object – the number 4, say – and study it exclusively. Of course, the utility in this would be dubious, so in practice no one does this. Instead, mathematicians identify collections of objects which share some common properties, and study the *collection* as a whole. The kinds of things they study depends entirely on the collection they are studying, of course; but generally, the hope is to gain some kind of broad understanding by looking at several objects at once, focusing on the properties these objects have in common.

To help make all of this rigorous enough to live up to the standards of modern mathematics, mathematicians have devised a language for describing such collections. We will look at a simplified version of this language.

A *set* is a collection of objects. In mathematics the objects are always abstract, like the "set of all positive numbers," or "the set of all real analytic functions[1]." In order to communicate their ideas, mathematicians have devised a notation for describing sets. In writing, there are three standard ways to describe a set:

(1) Some sets are so well-known in mathematics that they have standardized names. These include $\mathbb{R}, \mathbb{Z}, \mathbb{C}, \mathbb{N}, \mathrm{GL}_n(k)$, to name a few. You probably haven't heard of these yet; we'll look at some of them later.
(2) Sometimes you can just write down a list of the objects in your set. Traditionally the list is enclosed in braces. For instance, the following are two valid examples of

[1]This is just an example; we won't be working with these in our class

how to write down a set:

$$\{1, 3, 76, 2, 4\},$$

$$\{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}.$$

(3) Sometimes it would be difficult to list the objects in your set explicitly, even though you could describe some property that they all have in common. In this case, the notation looks a little silly, but it does follow a pattern. Here are some examples:

$$\{x : x \text{ is an odd number}\},$$

$$\{x + y : x \text{ is between 0 and 1 and } y \text{ is a multiple of } 3/5\}.$$

Here the first example is just the set of all odd numbers. The second example is a bit more complicated: it says that, if x is between 0 and 1, and y is a multiple of $3/5$, then the number $x + y$ is in the set. As we can see in the second case, it would be hard to just list all of the numbers in the set, but a description of the set is otherwise easy to write down.

Which notation you use depends on how much you know about the set and what you plan to do with it.

Before we progress, there is an important rule to keep in mind about sets: a set only keeps track of what objects are in it, not *how many copies* of each object there are. A question like "how many times does the number 2 appear in this set?" has no meaning. For instance, the following two sets are *exactly the same*, since they have the same objects in them, even though they have been listed a different number of times:

$$\{1, 2, 3, 3, 4, 5\},$$

$$\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 4, 5, 5, 5, 5\},$$

$$\{1, 2, 3, 4, 5\}.$$

This rule is just a convention, of course; mathematicians didn't *need* to adopt this convention, but (for fairly technical reasons) this rule actually helps to simplify a lot of mathematical works.

Another thing to keep in mind is that sets have no notion of *order*. If we think of the objects as being ordered, that information is kept somewhere else *outside* of the set. So for instance, the following sets are the same:

$$\{1, 2, 3\},$$

$$\{1, 3, 2\},$$

$$\{2, 3, 1\}.$$

Sure, *we* know that $1 < 2 < 3$, but the *set* doesn't know this.

When two sets have some objects in common, we say that the sets *intersect*. When two sets have no objects in common, we say that the sets are *disjoint*. For example:

$$\{1, 2, 3\} \text{ and } \{2, 3, 4\} \text{ intersect,}$$

$$\{1, 2, 3\} \text{ and } \{7, 8, 9\} \text{ are disjoint.}$$

As mentioned earlier, some sets are so important that they have been given standard names. This is especially common when talking about sets of numbers. Here are some examples of named sets of numbers:

- $\mathbb{N}$, the set of counting numbers, or *natural numbers*. $\mathbb{N}$ is defined to be the set
$$\mathbb{N} = \{0, 1, 2, 3, 4, ...\}.$$
$\mathbb{N}$ is an example of a *semiring*: that is, if you start with two numbers in $\mathbb{N}$ and you add or multiply them together, the result is also in $\mathbb{N}$. For instance, 3 and 7 is in $\mathbb{N}$, as is $3 + 7$ and 3×7. If you grab two numbers in $\mathbb{N}$, sometimes you can subtract or divide them and get another number in $\mathbb{N}$, but not always.
- $\mathbb{Z}$ is the set of all whole numbers, or *integers*[2]. $\mathbb{Z}$ is defined to be the set
$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}.$$
$\mathbb{Z}$ is an example of a *ring*: that is, if you start with two numbers in $\mathbb{Z}$ and you add, subtract, or multiply them, the result will also be in $\mathbb{Z}$. Division, however, is still a mixed bag.
- $\mathbb{Q}$ is the set of fractions, or *rational numbers*. By definition, they include all of $\mathbb{Z}$, since every integer is also a fraction (the number n is equal to the fraction $\frac{n}{1}$). $\mathbb{Q}$ is defined to be the set
$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \text{ are in } \mathbb{Z} \text{ and } b \neq 0 \right\}.$$
$\mathbb{Q}$ is an example of a *field*: that is, if you start with two numbers in $\mathbb{Q}$ and you add, subtract, multiply, or divide them, the result will always be in $\mathbb{Q}$ as well (unless you try to divide by 0, in which case the result is simply undefined).
- $\mathbb{R}$ is the set of *real numbers*. It's definition is technical[3], but it has an intuitive definition: $\mathbb{R}$ is the set of all numbers (except for the imaginary numbers). This means that $\mathbb{R}$ contains all of $\mathbb{Q}$, in addition to numbers not in $\mathbb{Q}$ (such as π and e). $\mathbb{R}$ is also a field.
- $\mathbb{C}$ is the set of *complex numbers*, also called the imaginary numbers. $\mathbb{C}$ is also a field. There are many ways to define $\mathbb{C}$. Here is one that is reasonably direct, if you are comfortable with the notation:
$$\mathbb{C} = \{x + iy : x, y \text{ are in } \mathbb{R}\},$$

[2] The symbol $\mathbb{Z}$ is used because the German word for "integer" starts with a "z."
[3] It is defined to be the set of all equivalence classes of Cauchy sequences in $\mathbb{Q}$.

and where $i = \sqrt{-1}$. Algebraically we can also think of $\mathbb{C}$ as the *algebraic closure* of $\mathbb{R}$, though you probably have not yet been introduced to this terminology. There's another algebraic definition which is likely to be equally foreign, but I write it here because the notation looks neat:

$$\mathbb{C} = \mathbb{R}[x]/(x^2 + 1).$$

- $\mathbb{S}^1$ and $\mathbb{S}^2$, the set of points in the circle and the sphere, respectively. We can define these sets as

  $$\mathbb{S}^1 = \left\{(x, y) : x^2 + y^2 = 1\right\} \text{ and } \mathbb{S}^2 = \left\{(x, y, z) : x^2 + y^2 + z^2 = 1\right\}.$$

  We'll talk more about $\mathbb{S}^1$ and $\mathbb{S}^2$ later in the semester, though we aren't going to be so technical about it when we get there.

There are lots of other sets as well, of course, but hopefully this is enough examples for now.

2. Arguments

Professional mathematicians are in the business of *identifying* properties of mathematical objects, usually through some combination of intuition and tinkering with examples, and then *proving* that these properties really do hold in general (that is, proving that the properties they identified weren't merely coincidences or the result of some mistake). The mathematics community has been doing this for over 2000 years. Over the years there have been changes in the standards that determines what a valid mathematical argument is; over time, however, the changes have all been made to increase the reliability of the results. A great deal of work in the 19th and 20th centuries has been devoted to re-evaluating these standards, ensuring that the work that our generation of mathematicians contributes will remain useful in the future.

The rules for a mathematical argument are *simple*, but this does not mean they are *easy*. Young mathematicians typically spend a semester learning the basic rules. We don't have that much time, so we will learn a (very) simplified version.

An argument always starts with some hypothesis (typically called a *premise*) and ends with a conclusion. There are two types of arguments, differing in how they connect the premise with the conclusion:

(1) *deductive arguments*, which start with a very broad premise and make a very specific conclusion, and
(2) *inductive arguments*, which start with a set of examples and conclude that they indicate a bigger pattern.

Unfortunately, the sames do little to help us remember which is which; it seems that one simply needs to use the language for a while before the terminology sticks.

Let's look at some examples. Here is a deductive argument:

(P1) Every state is smaller than Alaska.
(P2) Montana is a state.
(C1) Therefore Montana is smaller than Alaska.

In this argument, the premise consists of lines (P1) and (P2), while the conclusion is just the line (C1). This is a deductive argument because it hinges on the *broad* fact that "every state is smaller than Alaska." This fact is broad in that it is a statement about a *set* of objects. (The premise (P2) does not influence the issue of whether or not this argument is inductive or deductive; the very existence of (P1) settles the issue.)

Here is an example of an inductive argument:

(P1) Barry Goldwater was a senator from Arizona who lost a presidential race.
(P2) John McCain is a senator from Arizona who lost a presidential race.
(C1) Therefore no senators from Arizona will ever be president.

This argument is inductive because it starts with a collection of examples and argues that they are part of a larger pattern.

To be more precise:

- A *deductive* argument begins with a statement about a set (for instance, the set of all states), then applies this statement to an object in that set (for instance, the state of Montana).
- An *inductive* argument begins with several statements about objects in a set (for instance, senators Goldwater and McCain) and concludes that the statements must be true for the entire set (for instance, the set of all senators from Arizona).

When presented with an argument, our first reaction should be to judge its quality. Inductive and deductive arguments are judged differently, but there are still things we look for in each.

A deductive argument is said to be *valid* if the conclusion really does follow from the premise; it is said to be *sound* if the conclusion follows from the premise *and* the premise is true.

Our example about Montana and Alaska is an example of a sound argument. Here is an argument which is valid but not sound:

> Everyone who owns a lightsaber is awesome.
> Tim owns a lightsaber.
> Therefore Tim is awesome.

It is decidedly true that everyone with a lightsaber is awesome. So, granting the premise that Tim has a lightsaber, he must be awsome – and thus the argument is valid. Unfortunately, it is not sound: Tim does *not* own a lightsaber.

Here is an example of an argument which is neither valid nor sound:

> The sky is blue.
> Tim likes blue things.
> Therefore Tim has a pet cat.

Since the conclusion does not follow from the premise, the argument is neither valid nor sound. The fact that the premise happens to be true is irrelevant.

In principle, given a deductive argument it should be completely clear if it is valid, sound, or neither. In reality, of course, some facts are disputed (such as matters of opinion, like "All bands are worse than Stix"), so in practice it may be difficult to assess the soundness and validity of arguments. Still, for arguments in an objective setting (like mathematics), it should always be possible to judge a deductive argument according to its soundness and validity.

Inductive arguments, on the other hand, are highly subjective. In an inductive argument, we attempt to assess the *strength* of the argument based on whether or not we believe the examples provided truly are representative of the big picture.

For instance, suppose there where $10,000$ people at a concert. Here is an example of a strong inductive argument:

> $9,943$ people said the concert was awesome.
> Therefore it was awesome.

This argument provides $9,943$ examples of people who thought the concert was awesome. Pretty convincing.

For comparison, here is an example of a weak inductive argument:

> Tim is a mathematician who can ride a unicycle.
> Dylan is a mathematician who can ride a unicycle.
> Ron is a mathematician who can ride a unicycle.
> Therefore all mathematicians can ride a unicycle.

Three people is hardly enough examples for us to make a conclusion about *all* mathematicians, so this argument is quite weak.

In mathematics, all arguments are deductive in their core, though the language and style used to express the ideas can seem to hide this fact from a person not accustomed to reading such things. What's worse is that mathematicians use the word "induction" *differently* than what we've just discussed! Indeed, since the type of induction we just described is so obviously subjective, that style of reasoning would never be allowed in a mathematical work. *Mathematical induction* – which mathematicians just call "induction" – is a technical trick which *appears* to use inductive reasoning, but is actually deductive at its core.

We won't be dealing with mathematical induction in our course; indeed, new mathematicians often need some time to get accustomed to it, since it *is* a trick.

3. An example of a mathematical proof

Our discussion wouldn't be complete without an example of a mathematical proof. This proof is due to Euclid, and is over 2000 years old. His presentation was undoubtedly quite different, but the idea is his:

Recall that a whole number is called a *prime number* if it is only divisible by 1 and itself. For instance, the numbers $2, 3, 5, 7, 11, 13, 17, 19, 23$ are all prime, while the numbers $4, 6, 18, 100$ are not. Obviously, every number is either prime or it is not prime. In fact, every number is either prime *or* it is divisible by a prime number.

When one identifies a special kind of number, the first question is "how many are there?" For prime numbers, the answer is known:

Theorem 1 (Euclid)**.** *There are infinitely-many prime numbers.*

Proof. Suppose this is false. In that case, there are only finitely-many prime numbers. For the sake of argument, let us say there are only k prime numbers. List them as $p_1, p_2, p_3, ..., p_k$.

Let a be the number defined by

$$a = p_1 \cdot p_2 \cdot p_3 \cdot \cdots \cdot p_{k-1} \cdot p_k + 1.$$

That is, a is just the product of all of the prime numbers, plus one.

We now ask: is a a prime number? By the way we defined a, it is obvious that $a \neq p_1$ and $a \neq p_2$ and so forth, since a is bigger than each of these numbers. Since $p_1, p_2, ..., p_k$ are the only primes, and a is not one of them, it follows that a is *not* a prime number.

Since a is not prime, it must be divisible by one of the prime numbers. For the sake of argument, suppose it is divisible by p_1 (it could be divisible by p_2 or p_5 or whatever; the details of which one doesn't change our argument).

Since a is divisible by p_1, the number $\frac{a}{p_1}$ is a whole number. But this can't be true! Because by the way we defined a, we know that

$$\begin{aligned} \frac{a}{p_1} &= \frac{p_1 \cdot p_2 \cdot p_3 \cdot \cdots \cdot p_k \cdot p_k + 1}{p_1} \\ &= \frac{p_1 \cdot p_2 \cdot p_3 \cdot \cdots \cdot p_k \cdot p_k}{p_1} + \frac{1}{p_1} \\ &= p_2 \cdot p_3 \cdot \cdots \cdot p_k \cdot p_k + \frac{1}{p_1} \end{aligned}$$

and thus

$$\frac{a}{p_1} - p_2 \cdot p_3 \cdot \cdots \cdot p_k = \frac{1}{p_1}.$$

Since $\frac{a}{p_1}$ and $p_2 \cdot p_3 \cdot \cdots \cdot p_k$ are both whole numbers, it follows that $\frac{a}{p_1} - p_2 \cdot p_3 \cdot \cdots \cdot p_k$ is a whole number. But this difference is equal to $\frac{1}{p_1}$, which is *not* a whole number. We have thus arrived at a contradiction, which means one of our assumptions was incorrect. Since our only assumption was that there are only finitely-many prime numbers, we conclude that there are, in fact, infinitely many prime numbers. $\square$

Of course, when mathematicians write for one another they are not usually so verbose; but the idea is the same.