

EXPONENTIAL CHANGE

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1. INTRODUCTION

Previously we looked at some examples of linear and exponential change, then delved into some of the details about linear change. Today we are going to look further into exponential change.

2. EXPONENTIAL CHANGE

As discussed earlier, a quantity has exponential change if the amount of change it undergoes a *relative* change each time period. Functions modelling exponential change always have the form

$$y = q_0(r^x),$$

where x is some amount of time, r is the rate of change, q_0 is the initial value of the quantity, and y is the value at time x .

We've seen several examples of exponential change already. Exponential change usually presents itself as a change by some percentage or factor in each unit of time. This is demonstrated in statements like "the cost of milk goes up by 3% annually," or "the time it takes to fly across the Pacific ocean decreases by a factor of 0.98 every three years."

Today we are simply going to example more examples of exponential change, and the kinds of things associated with it.

3. DOUBLING TIME AND HALF LIFE

In a system undergoing exponential growth, the time it takes for a quantity to double in size is the system's *doubling time*. In a system undergoing exponential decay, the time it takes for a quantity to be cut in half is the system's *half life*.

Doubling time/half life says a lot about a system. Let's look at how these ideas interact with what we've done so far.

First, a formula: if a system's doubling time is T , and its initial value is q_0 , then the formula for the system's growth is

$$y = q_0 2^{t/T},$$

where y is the value of the quantity at time t . Similarly, if the system's half life is T , and its initial value is q_0 , then the formula for the system's decay is

$$y = q_0 \left(\frac{1}{2}\right)^{t/T}.$$

Here's an example. Suppose that the world's population doubles every 40 years. IT was 6 billion in 2000. What will it be in 2030? In 2200? To answer this, we need only use the formula:

$$\begin{aligned} \text{Population in 2030} &= 6,000,000,000 \times 2^{30/40} = 10,090,756,983, \\ \text{Population in 2200} &= 6,000,000,000 \times 2^{200/40} = 192,000,000,000. \end{aligned}$$

Here's another example. Suppose that your bank account doubles in size every 13 years. By what factor does your balance increase in 50 years? To compute, we simply plug in to the formula:

$$2^{50/13} = 14.38.$$

Sometimes we are told that a system is growing (or shrinking) by some percentage every period of time (say, every year, or every minute). In such situations, we may be asked what the system's doubling time (or half life) is. If the growth (or decay) is less than 15% per unit of time, there is a handy approximation that works surprisingly well: for a system undergoing change by $P\%$ per unit of time, the doubling time (or half life) is approximately

$$\frac{70}{P} \text{ units of time.}$$

For instance, if the cost of milk goes up by 2% every year, then the doubling time for the cost of milk is

$$\frac{70}{2} = 35 \text{ years.}$$

Here's another example, this time dealing with decay. Suppose that inflation causes the Russian ruble to fall at 12% per year relative to the US dollar. How long does it take to lose

half of its value? We just plug in the numbers:

$$\frac{70}{12} = 5.83 \text{ years.}$$

When the rate of change is greater than 15% per time unit, these approximations start to break down pretty quickly. In such situations, we would do well to use more precise formulas. In a system growing at a rate of $R\%$ per time unit, the doubling time is precisely

$$\text{doubling time} = \frac{\log(2)}{\log(1 + R/100)} \text{ time units.}$$

In a system shrinking at a rate of $R\%$ per time unit, the half life is precisely

$$\text{half life} = \frac{\log(1/2)}{\log(1 - R/100)} \text{ time units.}$$

For instance, the *precise* half life in our Russian ruble example would have been

$$\frac{\log(1/2)}{\log(1 - 0.12)} = 5.42 \text{ years.}$$

4. WHY DOES THE APPROXIMATION WORK?

In Figure (1) I've graphed the functions

$$y = \frac{70}{x} \text{ and } y = \frac{\log(1/2)}{\log(1 - x/100)},$$

which are the functions for the approximate and precise half lives, respectively. If you look in the figure, the curves are very close to one another for values of $x < 15$, while for values of $x > 15$ the curves quickly diverge from one another. The fact that these functions happen to be so close together is why $\frac{70}{x}$ is such a good approximation for small values of x .

Approximations like this are relatively easy to produce, if one is familiar with techniques from Calculus. For us, we may simply use the approximation without being concerned with how it was developed (this, sadly, must be saved for another class).

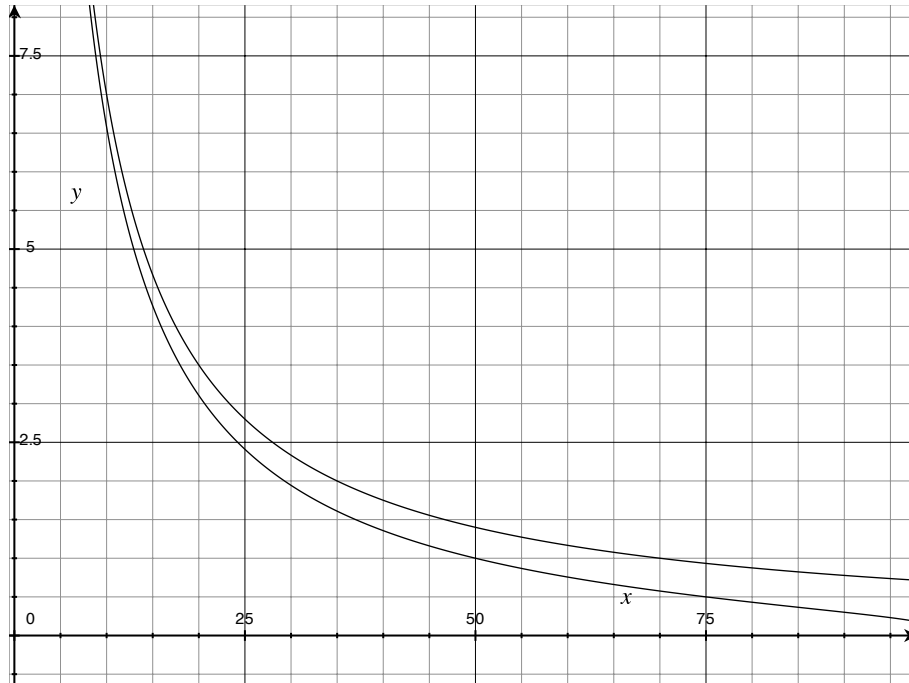


FIGURE 1. Graphs of the exact and approximate half life functions. The approximation is the lower curve. Notice that, for values of $x < 15$, the curves are very close together.