FINANCE

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1. Interest

People loan money to each other all the time. Obviously when you use a credit card, or take out a student or auto loan, someone is loaning you money. When you deposit money into the bank, you are loaning money to the bank. Usually when someone loans someone else money, there is an understanding that the money will be paid back *with interest*.

When it comes to interest, there are two common types, *simple interest* and *compounding interest*.

1.1. Simple Interest.

Suppose you loan someone \$100, with the agreement that they will pay you back in 20 weeks, at which time they will pay you back your \$100 plus an extra \$2 per week. This is an example of *simple interest*. With simple interest, the amount of interest generated is determined only by the initial amount of money, and does not compound.

Here's another example. Suppose someone loans you \$1200, with 2.4% simple interest accumulating monthly. How much money will you owe if you wait 7 years to pay off the loan? To figure this out, we just need to figure out how much interest will have accumulated in 7 months, and add that to the principal (the initial amount you borrowed). Mathematically (note that we keep track of the units!),

$$\begin{aligned} 7\text{months} \times 2.4\%/\text{month} \times \$1200 + \$1200 &= 7 \times 0.024 \times \$1200 + \$1200 \\ &= \$201.60 + \$1200 \\ &= \$1401.60. \end{aligned}$$

We can take this example and use it to build a formula. If an account has a principal balance of P dollars, and earns simple interest $R\%$ accumulating n times per year, then after y years the balance of the account would be

$$\text{Balance after } y \text{ years} = \underbrace{\frac{R}{100} \times P \times n \times y}_{\text{interest accrued}} + \underbrace{P}_{\text{principal}}.$$

In the above example, $P = \$1200$, $R = 2.4$, $n = 12$, and $y = \frac{7}{12}$.

Here's another example. You loan someone \$876 with 6% simple interest accumulating monthly. How much will they owe you if they wait 2 years to pay off the loan? For this, we can simply plug the numbers into the formula:

$$\frac{6}{100} \times 12 \times 2 \times \$876 + \$876 = \$2,137.44.$$

1.2. **Compounding interest.** *Compounding interest* is a scheme in which an account or loan accumulates "interest on interest." The idea is that each time interest is paid out, the *amount* of interest paid is determined by both the initial balance *and* the interest which has already accumulated. In bank accounts and loans, compounding interest is by far the more common type.

Let's look at an example of compounding interest. Suppose you have a bank account which offers 2.2% compounding interest annually. If you put \$100 in the account, how much money will be in the account in 3 years?

To answer this, we can build a table, listing the balance after 1, 2, and finally 3 years:

No. of years	Balance
0	\$100.00
1	$\$100.00 + 0.022 \times \$100.00 = \$102.20$
2	$\$102.20 + 0.022 \times \$102.20 = \$104.45$
3	$\$104.45 + 0.022 \times \$104.45 = \$106.75$

Alternatively, we could have used the following formula to perform the calculation. In an account with an initial balance of P dollars, with an *annual percentage rate* of $R\%$, in which interest is paid out n times per year, the balance after y years is given by

$$\text{Balance after } y \text{ years} = P\left(1 + \frac{R/100}{n}\right)^{ny}.$$

So in the above example, $P = \$100$, $R = 2.2$, $n = 1$, and $y = 3$.

Here's another example. Suppose you have \$9,700 in an account with an annual percentage rate of 4.3% compounding monthly. How much will be in the account in 100 years? To arrive at the answer, we simply use the formula:

$$\$9700\left(1 + \frac{0.043}{12}\right)^{12\times100} = \$709,414.57.$$

1.3. **Waiting for more money.** Suppose you put some money into an account with compounding interest. Given the annual percentage rate, and knowing how often per year the account gains interest, could you figure out how long it will take for your money to double? Triple?

The answer is *yes*. Let's look at the compounding interest formula:

$$\text{Balance after } y \text{ years} = P \underbrace{\left(1 + \frac{R/100}{n}\right)^{ny}}_{\text{Growth factor}}.$$

Let's say you want to know how long it will take for your money to double. What you're really saying is that you want to find y so that

$$\text{Balance after } y \text{ years} = 2P.$$

We can use this to setup an equation that we can solve for y:

$$\begin{aligned}
2P &= P\left(1 + \frac{R/100}{n}\right)^{ny} & & \\
2 &= \left(1 + \frac{R/100}{n}\right)^{ny} & & \text{Cancel the } P\text{'s} \\
\log 2 &= \log\left[\left(1 + \frac{R/100}{n}\right)^{ny}\right] & & \text{Take the log of both sides} \\
\log 2 &= ny \log\left(1 + \frac{R/100}{n}\right) & & \text{Use the exponent property of logs} \\
\frac{\log 2}{n \log\left(1 + \frac{R/100}{n}\right)} &= y & & \text{Move the non-}y\text{ stuff to the other side.}
\end{aligned}$$

For instance, let's say I have an account which earns 3.56% interest compounding monthly. How long will it take for money in this account to double? I just plug the numbers into the formula above: $R = 3.56$, $n = 12$, and get

$$\frac{\log 2}{12 \log\left(1 + \frac{3.56/100}{12}\right)} = 19.5 = y,$$

where y (as usual) is measured in years.

2. Paying off loans

Typically when a person takes out a loan they don't pay it off all at once; instead, they make regular payments over time until the loan is paid off. Suppose we have a loan that we want to pay off in some number of years, making some number of payments each year, with each payment being the same size as the rest. How would we calculate the amount of money to include in each payment?

There's a formula for answering this question. The formula looks complicated, but using it isn't especially difficult. Here it is: if P is the principal (the amount of money initially loaned), R is the interest rate (as a percentage), and we want to pay off the loan in y years making n payments per year, then

$$\text{Payment} = \frac{P\left(\frac{R/100}{n}\right)}{1-\left(1+\frac{R/100}{n}\right)^{-ny}}.$$

This formula has some assumptions built in:

(1) We want each of our payments to be of the same size.
(2) The loan has compounding interest.
(3) The interest compounds the same number of times per year as our payments are due. (For instance, if the interest compounds monthly, then the formula assumes we want to make monthly payments. This formula could not handle a situation where, for instance, the interest compounds annually but we wanted to make weekly payments.)

Here is a sample problem. Suppose you want to pay off a loan for $\$10,000$ in 2 years, and the loan has 2.2% interest compounding monthly. How large would your monthly payments be? We just plug the numbers into the formula:

$$\begin{aligned} \text{Payment} &= \frac{P\left(\frac{R/100}{n}\right)}{1-\left(1+\frac{R/100}{n}\right)^{-ny}} \\ &= \frac{\$10000\left(\frac{2.2/100}{12}\right)}{1-\left(1+\frac{2.2/100}{12}\right)^{-(12)(2)}} \\ &= \$426.28. \end{aligned}$$

When using this formula, if you are computing the answer in several steps (instead of just plugging the whole thing into your calculator all at once), it is *vital* that you not round the numbers along the way! If you round the numbers during your calculations, the answer can come out to be wildly different.

3. Saving up money

Let's now look at the opposite of loan payments – savings. Suppose you have a bank account that you make regular deposits into for some period of time. How would you compute what the total balance of the account will be at some point in the future? We have a formula for that, too. Let D be the amount of money you plan to deposit n times per year for y years. Let's say that the account has an annual interest rate of R, measures as a percent. Then after y years of this, the balance in the account will be

$$\text{Balance after } y \text{ years} = D\frac{\left(1+\frac{R/100}{n}\right)^{ny}-1}{\left(\frac{R/100}{n}\right)}.$$

Notice that this formula looks *similar* to the loan payment formula, but is certainly not the same. Still, it operates under similar assumptions:

(1) We want each of our deposits to be of the same size.
(2) The account has compounding interest.
(3) The interest compounds the same number of times per year as we make deposits. (For instance, if the interest compounds monthly, then the formula assumes we want to make monthly deposits. This formula could not handle a situation where, for instance, the interest compounds annually but we wanted to make weekly deposits.)

Let's look at a quick example. Suppose we want to make monthly deposits of \$120 for 9 months into an account with an interest rate of 3.4% compounding monthly. Then after 9 months (9 months is 0.75 years) the balance in the account will be

$$\begin{aligned}\text{Balance} &= D\frac{\left(1+\frac{R/100}{n}\right)^{ny}-1}{\left(\frac{R/100}{n}\right)}\\ &= \$120\frac{\left(1+\frac{3.4/100}{12}\right)^{(12)(0.75)}-1}{\left(\frac{3.4/100}{12}\right)}\\ &= \$1092.32.\end{aligned}$$