MATHEMATICAL MODELING

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1. Introduction

In this lecture, we are going to look at some first examples of how to use mathematics to model so-called "real world" quantities. Of course, many people make their entire career out of this sort of thing, and as such there's quite a bit one can say about the subject. Our time is limited, so we're going to look at some techniques which are both useful and simple.

2. Linear and Exponential Change

For now, we will limit our attention to modeling systems whose quantities change in time. With this in mind, we are now prepared to introduce our two main definitions:

Linear change occurs when a quantity changes by the same *fixed* amount in each unit of time. *Exponential change* occurs when a quantity changes by the same *relative* amount in each unit of time.

When we say "fixed amount," we mean some concrete, unchanging quantity. When we say "relative amount," we mean some percentage of the previous value of the quantity. For instance, consider two fictitious cities, one whose population grows by 22 people per year, the other whose population grows by 2.4% per year. The first is an example of linear change; the second is an example of exponential change.

Let's look at some examples. In each example, we'll decide if the change being described is linear or exponential.

(1) The size of East High has increased by 50 students each year. Five years ago it was 874. What is it now?

The change is by a fixed amount (50 students) so this must be linear change. To compute the answer, we just add 50 students for each year in question:

$$874 + \underbrace{50 + 50 + 50 + 50 + 50}_{\text{5 years}} = 1124 \text{ students.}$$

(2) The price of milk goes up by 3% per year. It was \$2 per gallon a year ago. What will it be in 2 years?

The change is by a relative amount (3% per year) so this must be exponential change. To compute the answer, we need to multiply by $(1+0.03)$ once for each year in question:

$$\$2\underbrace{(1+0.03)(1+0.03)(1+0.03)}_{3\text{ years}} = \$2.19.$$

(3) You have some equipment that was worth \$1000 three years ago. It depreciates by \$200 per year. What's it worth now?

The change is by a fixed amount (\$200 per year) so this must be linear change. Once again, we just subtract off \$200 for each year in question:

$$\$1000 - \underbrace{\$200 - \$200 - \$200}_{3\text{ years}} = \$400.$$

(4) Computer hard drives double in size every two years. Top of the line today is about 1.5 TB. What will it be in 5 years?

Since the size doubles every two years, it is increasing by a factor of 2 every two years, which is a factor of $\sqrt{2}$ every year[1]. Even though the problem does not use the percent symbol (%), this is still exponential change: this growth factor is a relative amount. If you like, we can think of it as growth by 141.42% per year. In any case, to compute the answer we simply multiply by $\sqrt{2}$ for each year in question:

$$1.5\underbrace{\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}}_{5\text{ years}} = 8.5\text{ TB}.$$

(5) HDTV costs drop 25% annually. Cost is \$1200 today. What will it be in 2 years?

Once more we have a relative change (25% per year) so we have exponential change on our hands. To get the answer we just multiply by $(1-0.25)$ for each year in question:

$$\$1200\underbrace{(1-0.25)(1-0.25)}_{2\text{ years}} = \$675.$$

These examples illustrate the two basic formulas that we'll be using. In a system with linear change, whose initial value is q_0 and which changes by a fixed amount c for every unit of time, the value of the quantity after t units of time is given by

$$q = q_0 + ct.$$

For instance, in Example (1) we had $q_0 = 874$, $c = 50$, and $t = 5$.

[1]How did I know this? The sizr changes by a factor of 2 every two years. I wanted to know the growth factor per year. So I setup the equation: $2 = x^2$, where x is the annual growth factor. Clearly $x = \sqrt{2}$.

In a system with exponential change, whose initial value is q_0 and which changes by a factor of r each unit of time, the value of the quantity after t units of time is given by

$$q = q_0(r^t).$$

For instance, in Example (2) we had $r = 1 + 0.03$.

The hardest part about doing modeling problems is figuring out if the change is linear or exponential, and then figuring out what values to put into the formulas.

3. More about linear change

In general, models are all about two things:

(1) Some initial data, such as
  (a) The number of dollars in my piggy bank
  (b) The amount of money on a gift card
  (c) The amount of gold in them thar hills
(2) The rate at which the data changes
  (a) My weekly allowance
  (b) Cost of a song on iTunes
  (c) Amount of gold mined every week

A linear model always takes the form

$$y = mx + b,$$

where x is the "input" and y is the "output." The value m is called the *slope*, while b is called the *y-intercept*. Earlier we had written this as

$$q = q_0 + ct,$$

which is really the same thing but with different letters.

If we do our job correctly, we can plug units into the modeling equation and the right units will come out. Let's look at some examples:

(1) We hiked a 3 mile trail with an initial elevation of 8000 feet. The trail gains 650 feet of elevation per mile. What was our elevation at the end of the hike?

$$3\,\text{mi}\,\frac{650\,\text{ft}}{\text{mi}} + 8000\,\text{ft} = 9950\,\text{ft}\,.$$

Here we have

$$\begin{aligned} x &= 3\,\mathrm{mi}\,, \\ m &= \frac{650\,\mathrm{ft}}{\mathrm{mi}}, \\ b &= 8000\,\mathrm{ft}\,, \\ y &= 9950\,\mathrm{ft}\,. \end{aligned}$$

(2) At \$2 each, a store sells 80 pineapplers per day. At \$5 each, the store sells 50 per day. Assuming a linear relationship, find an equation that predicts how many pineapples they will sell at a given price.

In this problem we have been given two pairs of x, y values, but we don't know m or b. Luckily we can setup a system of equations to find these values:

$$\begin{aligned} 80 &= m2 + b, \\ 50 &= m5 + b. \end{aligned}$$

We can solve this system like so:

$$\begin{aligned} 80 - 50 &= (m2 + b) - (m5 + b) \\ 30 &= -3m \\ -10 &= m \end{aligned}$$

which we can then plug back into either one of the original equations to get

$$\begin{aligned} 80 &= -20 + b \\ 100 &= b \end{aligned}$$

so that our equation is

$$y = -10x + 100,$$

where x is the price per pineapple and y is the number of pineapples sold per day at that price.