PERCENTAGES

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1. Percentages

There are many situations where we find ourselves trying to describe some fraction of a greater whole, such as "the fraction of students who live on campus," or "the portion of apples in the United States that were grown in Washington." In principle a fraction can be written with any denominator we like. For instance, the following fractions are all equal:

$$\frac{1}{2} = \frac{7}{14} = \frac{23.3}{46.6}.$$

In situations where we want to be able to draw comparisons between fractions, it is useful to standardize and write each of the fractions with denominator 100. In the above example, this would be the fraction $\frac{50}{100}$. Of course, if we write all of our fractions with denominator 100, then there's little sense in writing the denominator at all. Thus we have adopted the following notation:

$$x\% = \frac{x}{100}.$$

With this notation, the above fraction could have been written as 50%.

Here's an example: "in a newspaper survey, 65% of 1069 people said they think the president is doing a good job." How many people was that?

To answer the question, we setup the equation

$$\frac{65}{100} = \frac{x}{1069}$$

and solve for x, yielding $x = 694.85$, which we may round to 695.

In our class, we'll use percentages for three tasks:

(1) To describe a fraction, as in the newspaper example.
(2) To describe a change, such as the change in balance in a bank account.
(3) To make a comparsion, such as comparing the costs of different vehicles.

When thinking of percentages as describing groups in the same greater whole, here are three rules that guide us in our calculations:

(1) You can only add two percentages when the associated groups are disjoint. For instance, if X is the percentage of people in the United States who are teachers, and Y is the percentage of people in the United States who are engineers, then $X + Y$ is the percentage of people in the United States who are either a teacher or an engineer. On the other hand, if Z is the percentage of people in the United States who are students, it would not make sense to compute $X + Z$, since these two groups overlap (there are some students who are also teachers, after all).

(2) You can only subtract one percentage from another if the percentage you are subtracting corresponds to a subgroup. For instance, if X is the set of all men in the United States, and Y is the percentage of people in the United States who have been president, then $X - Y$ is the percentage of people in the United States who are men but have not ever been president.

(3) You can multiply percentages if the groups they describe line up in the right way. For instance, lets say we have three groups: A, B, and C, where C is contained in B, and B is contained in A.

   If X is the percent of objects from A which are in B, and Y is the percent of objects from B which are in C, then $X \times Y$ is the percent of objects from A which are n C. To do the calculation, however, we must be mindful to multiply X and Y together as fractions, not as the whole numbers we write them out as.

   For instance, Let A be the set of all freshmen, let B be the set of all freshmen who live in the dorm, and let C be the set of all freshmen women who live in the dorm. If $3/4 = 70\%$ of the freshmen live in the dorm, and if $2/3 = 33.3\%$ of the freshmen dorm dwellers are women, what percentage of the freshmen are women who live in the dorm?

   To answer this question, we simply multiply the percentages together:

$$75\% \times 33.3\% = \frac{3}{4} \times \frac{2}{3} = \frac{2}{4} = 50\%.$$

One last thing about computing with percentages: if you have a collection of percentages, in general it is a bad idea to try and "average" them. Here is an example from a famous lawsuit against the University of California, Berkeley.

> A lawsuit was filed against UC Berkeley for having a bias against women applying to the graduate school. The following table shows acceptance rates from 1973:

	Applicants	% Admitted
Men	8442	44%
Women	4321	35%

From this data, it appears as though there is a bias against women. However, if we break up the data by graduate department, the table looks quite different – in many cases, showing a bias against men:

Major	Men		Women	
	Applicants	% Admitted	Applicants	% Admitted
A	825	62%	108	82%
B	560	63%	25	68%
C	325	37%	593	34%
D	417	33%	375	35%
E	191	28%	393	24%
F	272	6%	341	7%

In this example, if we simply averaged the acceptance rates across the departments, we would *not* get the proper acceptance averages for the entire university.

2. Using percentages to describe relative change

Percentages provide us with a common language for describing change in *relative* terms. For instance, if I had \$7,000 in my bank account and I spent \$5, this is clearly a less significant change than if I had only \$22 in my bank account and I spent \$5. While qualitatively we know this to be true, percentages give us a language for describing this *quantitatively*.

When a quantity undergoes a change, the *percent change* is defined to be the ratio

$$\text{Percent change} = \frac{\text{New value} \; - \; \text{Old Value}}{\text{Old value}}.$$

So in the example above, the percent change in the balance of the larger account was only

$$\frac{6995 - 7000}{7000} = -0.07\%,$$

while in the smaller account the percent change was

$$\frac{17 - 22}{22} = -22.7\%.$$

3. Using percentages to relative differences

Lastly, percentages give us a way to describe the relative differences between two quantities. For instance, intuitively we may know that the difference between the price of a \$10,000 and a \$35,000 car is quite large, though for some reason the exact same cost difference between a $\$210,000$ and $\$235,000$ car does not seem so significant. Once again, percentages give us a language to make this intuition precise.

Given two values A and B, the *percent difference* between them is defined to be the ratio

$$\text{Percent change} = \frac{A-B}{B}.$$

Note that, in general, the ratios $\frac{A-B}{B}$ and $\frac{B-A}{A}$ will be *different numbers*. For this reason, we must introduce some extra language to remove any possible ambiguity. When computing a percent difference, the quantity which we place in the denominator is called the *reference* value. Any time we are comparing two numbers, we must first decide which to take as our reference.

In the above example, let's make the cheaper of the cars our reference value. For the working man's cars, the percent difference is

$$\frac{35,000 - 10,000}{10,000} = 250\%,$$

while for the super cars the percent difference is

$$\frac{235,000 - 210,000}{210,000} = 11.9\%.$$

Since we used the cheaper of the cars as our reference, we might interpret the first result by saying "The more expensive car costs 250% more than the less expensive car."

For the sake of experiment, lets swap our references, this time taking the more expensive of the cars as our reference. Then we obtain

$$\frac{10,000 - 35,000}{35,000} = -71.4\%,$$

$$\frac{210,000 - 235,000}{235,000} = -10.6\%,$$

which we might interpret (in the first case) by saying "The less expensive car costs 71.4% less than the more expensive car."