UNITS OF MEASUREMENT

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1. Units

Units provide us with a standard framework for comparing measurements. They can also be used as a bookkeeping method when making calculations about the physical world. In this lecture we will look at some common units and practice a technique of solving problems simply by manipulating the units involved.

Today there are two dominant systems of units: the *metric system* or "SI" (standing for "Système International d'Unités" in French), which is used in almost every country on earth, and the *standard* or *Imperial*[1], which is used in the United State, Myanmar, and Liberia, and in certain limited contexts elsewhere (such as British road signs).

There are some units we're all familiar with. *Feet*, *miles*, and *inches* are common units for measuring lengths and distances in the Imperial system. In the metric system, *meters* are used to describe distance. *Quarts*, *cups*, and *gallons* are used to describe volume in the Imperial system, while the metric system uses *liters.* We'll look at more examples of units as we work through some problems.

First, let's look at how calculations with units are performed. There are two rules:

(1) To add or subtract two quantities, they must be measured in the *same* units. For instance,
$$15\,\mathrm{ft} + 16\,\mathrm{ft} = 31\,\mathrm{ft}\,.$$
(2) We can always multiply or divide two quantities regardless of their units. The units come along for the ride. This is precisely why we have units like "miles per hour" (a measure of speed, mph $= \frac{\mathrm{mi}}{\mathrm{hr}}$) and "pounds per square inch" (a measure of pressure, psi $= \frac{\mathrm{lbs}}{\mathrm{in}^2}$). For instance,
$$72\,\mathrm{mph} \times 2\,\mathrm{hr} = 72\frac{\mathrm{mi}}{\mathrm{hr}} \times 2\,\mathrm{hr} = 144\,\mathrm{mi},$$

[1]The system has its origins in the British empire, though the United Kingdom now uses the metric system.

$$\frac{12\,\text{lbs}}{6\,\text{in}^2} = \frac{12}{6}\frac{\text{lbs}}{\text{in}^2} = 2\frac{\text{lbs}}{\text{in}^2} = 2\,\text{psi}\,.$$

Notice that these are *precisely* the same rules we have when simplifying expressions that contain variables. The first example could have been written as

$$15x + 16x = 31x$$

while the next example could have been

$$72\frac{x}{y} \cdot 2y = 144x.$$

Let's look at an example problem real quick:

Problem 1. *Tony traveled a distance of* 1232 *miles over a span of* 823 *minutes. What was his (average) speed?*

Speed is distance divided by time, so we compute that

$$\text{speed} = \frac{1232\,\text{mi}}{823\,\text{min}} = \frac{1232}{823}\frac{\text{mi}}{\text{min}} = 1.5\frac{\text{mi}}{\text{min}}.$$

2. The metric system

The metric system is a wonderful system of units. The system is based on a small collection of basic units, which are modified by a system of prefixes.

The base units:

Unit	Purpose
meter	Distance
liter	Volume
second	Time
gram	Mass

There are a few *derived* units in the metric system as well. The most common is the *Newton*, denoted N, which is used to measure force. It is defined by the relation $1\,\text{N} = 1\frac{\text{g}\cdot\text{m}}{\text{s}^2}$.

The prefixes:

Prefix	Multiplier
mega	10^6
kilo	10^3
hecto	10^2
deca	10^1
deci	10^{-1}
centi	10^{-2}
mlli	10^{-3}
micro	10^{-6}
nano	10^{-9}
pico	10^{-12}

For instance, a *kilometer* is 1000 m, while a *millimeter* is $\frac{1}{1000}$ m.

3. Converting units

Most of the problems we will encouter will require us to *convert* between different units. Unit conversion is necessary any time we have a quantity measured in one unit, but we wish that it were measured in another unit. It is possible any time the two units are used to measure the same physical attribute.

Let's do an example. One meter is 3.28 feet, or written symbolically, $1\,\mathrm{m} = 3.28\,\mathrm{ft}$. When converting units, we interpret this equality as literally as possible. In other words, since $1\,\mathrm{m} = 3.28\,\mathrm{ft}$, we may also write

$$1 = \frac{1\,\mathrm{m}}{3.28\,\mathrm{ft}} \text{ and } 1 = \frac{3.28\,\mathrm{ft}}{1\,\mathrm{m}}.$$

This gives us a "fancy way" of writing the number 1.

Remember that any number "times 1" yield the same number. This idea, together with the fractions written above, is the key to converting units. For instance, suppose I wish to convert 342 m into feet. No problem:

$$\begin{aligned}
342\,\mathrm{m} &= \Big(342\,\mathrm{m}\Big)\Big(1\Big) \\
&= \Big(342\,\mathrm{m}\Big)\Big(\frac{3.28\,\mathrm{ft}}{1\,\mathrm{m}}\Big) \\
&= \Big(\frac{342\,\mathrm{m}}{1}\Big)\Big(\frac{3.28\,\mathrm{ft}}{1\,\mathrm{m}}\Big) \\
&= \frac{342 \cdot 3.28}{1}\frac{\mathrm{m}\cdot\mathrm{ft}}{\mathrm{m}} \\
&= \frac{342 \cdot 3.28}{1}\frac{\mathrm{ft}}{1} \\
&= 342 \cdot 3.28\,\mathrm{ft} \\
&= 1121.76\,\mathrm{ft}\,.
\end{aligned}$$

Of course, if we didn't wish to be so verbose we may have simply written

$$342\,\mathrm{m} = (342\,\mathrm{m})(1) = \left(342\,\mathrm{m}\right)\left(\frac{3.28\,\mathrm{ft}}{1\,\mathrm{m}}\right) = 1121.76\,\mathrm{ft}\,, \tag{1}$$

provided we can do a little algebra in our heads.

In general, here is what a conversion problem looks like: you start with some quantity q in some unit $\mathbf{u}$. You wish it was measured in a different unit $\mathbf{w}$. You have some equation relating these units, say $a\,\mathbf{u} = b\,\mathbf{w}$. Here is how you do the conversion:

$$q\,\mathbf{w} = (q\,\mathbf{w})(1) = \Big(q\,\mathbf{w}\Big)\Big(\frac{a\,\mathbf{u}}{b\,\mathbf{w}}\Big) = \frac{q \cdot a}{b}\mathbf{u}.$$

Notice how this equation compares to Equation (1).

Of course, we can repeat this process many times over. For instance, suppose we wanted to convert 2 *days* into *seconds*. We know the following:

$$\begin{aligned}
60\,\mathrm{s} &= 1\,\mathrm{min} \\
60\,\mathrm{min} &= 1\,\mathrm{hr} \\
24\,\mathrm{hr} &= 1\,\mathrm{day}
\end{aligned}$$

so that

$$2\,\mathrm{days} = 2\,\mathrm{days}\left(\frac{24\,\mathrm{hr}}{1\,\mathrm{day}}\right)\left(\frac{60\,\mathrm{min}}{1\,\mathrm{hr}}\right)\left(\frac{60\,\mathrm{s}}{1\,\mathrm{min}}\right) = 172800\,\mathrm{s}\,.$$

4. More unit conversion

The previous section introduced unit conversion. In this section we will introduce an extra idea that can be used to trivialize many problems.

Consider the following problem:

Problem 2. 1 *gallon of gas costs* \$2.23. *Your car gets* 28 mi *to the gallon. Seattle is roughly* 850 mi *from here. How much would it cost you to drive one-way to Seattle?*

We could fiddle about, trying to remember where to multiply or divide. Instead, let's take a different approach: we have a measurement in miles (the distance 850 mi) and we want a measurement in dollars. Let's think of this as a unit conversion problem.

All we need to do is make the following interpretations

$$\begin{aligned} \text{“1 gallon of gas costs \$2.23”} \quad & \text{means} \quad 1 \text{ gallon of gas } = 2.23 \text{ dollars}, \\ \text{“Your car gets 28 mi to the gallon”} \quad & \text{means} \quad 28\,\text{mi} = 1 \text{ gallon of gas}. \end{aligned}$$

With these interpretations, the problem is easy:

$$850\,\text{mi} = 850\,\text{mi}\left(\frac{1 \text{ gallon of gas}}{28\,\text{mi}}\right)\left(\frac{2.23 \text{ dollars}}{1 \text{ gallon of gas}}\right) = 67.70 \text{ dollars}.$$

5. Power and energy

Lastly, we now discuss units of *power* and *energy*. First we must define our terms. *Energy* is the capacity to do work. It is what makes things move or heat up. The metric unit of energy is the *joule*. In the Imperial system, Calories and calories (yes, same word, different capitalization) are used to describe energy. The conversion is

$$1\,\text{Cal} = 4184\,\text{joule} \text{ and } 1\,\text{Cal} = 1000\,\text{cal}\,.$$

Power, on the other hand, is the rate at which energy is used. The metric unit of energy is the *watt*, which is defined as

$$1\,\text{watt} = 1\frac{\text{joule}}{\text{s}}.$$

Problem 3. *When riding a fitness bicycle, the readout states that you are burning* 500 *Calories per hour. Are you generating enough power to light a* 100 *watt lightbulb?*

To solve the problem we simply need to convert our energy usage from Calories per hour into watts. We do this with a chain of conversions:

$$500\frac{\text{Cal}}{\text{hr}} = \left(500\frac{\text{Cal}}{\text{hr}}\right)\left(\frac{4184\,\text{joule}}{1\,\text{Cal}}\right)\left(\frac{1\,\text{hr}}{60\,\text{min}}\right)\left(\frac{1\,\text{min}}{60\,\text{s}}\right) = 581\frac{\text{joule}}{\text{s}} = 581\,\text{watt}\,,$$

which is enough to light five 100 watt bulbs and then some.